Damped Simple Harmonic Motion

Pure simple harmonic motion\textsuperscript{1} is a sinusoidal motion, which is a theoretical form of motion since in all practical circumstances there is an element of friction or damping. A mechanical example of simple harmonic motion is illustrated in the following diagrams. A mass is attached to a spring as follows. This represents the equilibrium position.

The mass is then moved a distance $A$ in the $x$-direction as follows.

Consider the dynamics of the mass in a general position with displacement $x$ from the equilibrium position. The restoring force is due to the tension (or compression in the spring and is proportional to the displacement from the centre of motion. In the case of damping, we presume that the damping is proportional to the velocity so that the damping force is $c \dot{x}$, where $c$ is the damping coefficient.

\textsuperscript{1} \textit{Simple Harmonic Motion}
Applying Newton’s second law\textsuperscript{2}, force equals mass times acceleration, the force on the mass is the tension in the spring and is equal to \( kx \). This gives the equation

\[
m\ddot{x} = -kx - c\dot{x},
\]

since the tension (compression) and the damping act in the opposite direction to the displacement, giving the governing equation

\[
m\ddot{x} + c\dot{x} + kx = 0,
\]

which is an ordinary differential equation\textsuperscript{3}. Let us use Laplace transforms\textsuperscript{4} to solve the governing ordinary differential equation\textsuperscript{5}.

Let \( X(s) \) be the Laplace transform of \( x(t) \). Then the Laplace transform of \( \dot{x}(t) \) is \( sX(s) - x(0) \) and the Laplace transform of \( \ddot{x}(t) \) is \( s^2X(s) - s\dot{x}(0) - x(0) \), where \( x(0) \) and \( \dot{x}(0) \) are initial conditions, which are unspecified.

Substituting the Laplace transforms into the governing equation gives the following

\[
m(s^2X(s) - s\dot{x}(0) - \ddot{x}(0)) + c(sX(s) - x(0)) + kX(s) = 0.
\]

We now make \( X(s) \) the subject of the above equation, by first gathering together terms in \( X(s) \):

\[
(ms^2 + cs + k)X(s) = msx(0) + m\ddot{x}(0) + cx(0)
\]

to give

\[
X(s) = \frac{msx(0) + m\ddot{x}(0) + cx(0)}{(ms^2 + cs + k)}
\]

Looking at the table of Laplace transforms\textsuperscript{4}, there are a number of potential solutions, but we will need to manipulate the expression of \( X(s) \) in order to make progress.

\[
\begin{array}{|c|c|}
\hline
10 & e^{-at}t^n \\
\hline
11 & e^{-at}\sin(\omega t) \\
\hline
12 & e^{-at}\cos(\omega t) \\
\hline
13 & e^{-at}\cosh(\omega t) \\
\hline
14 & e^{-at}\sinh(\omega t) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
11 & \frac{n!}{(s + a)^{n+1}} \\
\hline
11 & \frac{\omega}{(s + a)^2 + \omega^2} \\
\hline
12 & \frac{s + a}{(s + a)^2 + \omega^2} \\
\hline
13 & \frac{s + a}{(s + a)^2 - \omega^2} \\
\hline
14 & \frac{\omega}{(s + a)^2 - \omega^2} \\
\hline
\end{array}
\]

\textsuperscript{2} Newton’s Laws of Motion

\textsuperscript{3} Ordinary Differential Equation

\textsuperscript{4} Laplace Transforms

\textsuperscript{5} Solving ODEs by Laplace Transforms
Since $s$ occurs on its own in the expressions in the table, the first step is to divide the terms through by $m$:

$$X(s) = \frac{sx(0) + \dot{x}(0) + \frac{c}{m} x(0)}{(s^2 + \frac{c}{m}s + \frac{k}{m})}. $$

In order to progress towards the "$(s+a)^2$" terms in all of the Laplace transforms in the table, we use the technique of completing the square. Using the fact that

$$(s + \frac{c}{2m})^2 = s^2 + \frac{c}{m}s + \frac{c^2}{4m^2},$$

then we can re-write the expression for $X(s)$ as follows:

$$X(s) = \frac{sx(0) + \dot{x}(0) + \frac{c}{m} x(0)}{(s + \frac{c}{2m})^2 + \frac{k}{m} - \frac{c^2}{4m^2}}.$$

Comparing this with the Laplace transform table, we note that $a = \frac{c}{2m}$. The term $\frac{k}{m} - \frac{c^2}{4m^2}$ may be positive if $c$ (the damping coefficient) is relatively small or negative if $c$ is relatively small or zero for a particular value for the damping coefficient $c = \sqrt{\frac{k}{m}}$. A term called the damping ratio $\zeta$ is introduced, such that $\zeta = \frac{c}{\sqrt{4km}}$. Let us consider four cases of the solution in turn.

**Undamped c=0 and $\zeta=0$**

It is useful to consider the undamped case, in which we return to simple harmonic motion. When $c=0$

$$X(s) = \frac{sx(0) + \dot{x}(0)}{(s^2 + \frac{k}{m})}. $$

From the table of Laplace transforms, in this special case we use

<table>
<thead>
<tr>
<th></th>
<th>$\sin(at)$</th>
<th>$\frac{a}{s^2 + a^2}$</th>
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<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\cos(at)$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
</tr>
</tbody>
</table>
In this case \( a = \sqrt{\frac{k}{m}} = \omega_n \), where \( \omega_n \) is the natural frequency. The solution is a sinusoidal with an angular frequency of \( \omega_n \).

**Underdamped** \( 0 < c < \sqrt{4km} \) or \( 0 < \zeta < 1 \)

When the damping is relatively small the system is said to be underdamped. In this case \( \frac{k}{m} - \frac{c^2}{4m^2} \) is positive.

\[
X(s) = \frac{sx(0) + \dot{x}(0) + \frac{c}{m}x(0)}{(s + \frac{c}{2m})^2 + \frac{k}{m} - \frac{c^2}{4m^2}} = \frac{sx(0) + \dot{x}(0) + \frac{c}{m}x(0)}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)).
\]

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<td>( e^{-at}\sin(\omega t) )</td>
<td>( \frac{\omega}{(s + a)^2 + \omega^2} )</td>
</tr>
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<td>( \frac{s + a}{(s + a)^2 + \omega^2} )</td>
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In which \( a = \frac{c}{2m} = \zeta \omega_n \) and \( \omega = \omega_n \sqrt{(1 - \zeta^2)} \). The solution consists of a sinusoid of angular frequency \( \omega \), which tends to \( \omega_n \) as the damping approaches zero, but approaches zero as the damping ratio approaches unity, or critical damping.

**Critical damping** \( c=\sqrt{4km} \) or \( \zeta=1 \)

In this case the Laplace transform of solution is as follows.

\[
X(s) = \frac{sx(0) + \dot{x}(0) + \frac{c}{m}x(0)}{(s + \omega_n)^2}
\]

This most closely matches

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<td>( \frac{n!}{(s + a)^{n+1}} )</td>
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</table>

in the table. Hence the solution is a linear combination of terms of the form \( e^{-\omega n t} \) and \( e^{-\omega n t} t \).

**Overdamping** \( c>\sqrt{4km} \) or \( \zeta>1 \)

In this case the Laplace transform of the solution has the following form
The form of the Laplace transform is closest to the following table item:

<table>
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<tr>
<th></th>
<th>$e^{-at}\cosh(\omega t)$</th>
<th>$\frac{s + a}{(s + a)^2 - \omega^2}$</th>
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<td>13</td>
<td></td>
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<td>$e^{-at}\sinh(\omega t)$</td>
<td>$\frac{\omega}{(s + a)^2 - \omega^2}$</td>
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where $a = \zeta \omega_n$ and $\omega = \omega_n \sqrt{\zeta^2 - 1}$. 